Exercise 21

Assume that the earth is a solid sphere of uniform density with mass M and radius $R = 3960$ mi. For a particle of mass m within the earth at a distance r from the earth's center, the gravitational force attracting the particle to the center is

$$F_r = \frac{-GM_r m}{r^2}$$

where G is the gravitational constant and M_r is the mass of the earth within the sphere of radius r.

(a) Show that $F_r = \dfrac{-GMm}{R^3} r$.

(b) Suppose a hole is drilled through the earth along a diameter. Show that if a particle of mass m is dropped from rest at the surface, into the hole, then the distance $y = y(t)$ of the particle from the center of the earth at time t is given by

$$y''(t) = -k^2 y(t)$$

where $k^2 = GM/R^3 = g/R$.

(c) Conclude from part (b) that the particle undergoes simple harmonic motion. Find the period T.

(d) With what speed does the particle pass through the center of the earth?

Solution

The mass of the earth is its density times its volume.

$$M = \rho\left(\frac{4}{3}R^3\right)$$

As a result, the earth's density is

$$\rho = \frac{M}{\frac{4}{3}R^3}.$$

The mass of earth within a sphere of radius r is

$$\begin{aligned} M_r &= \rho\left(\frac{4}{3}r^3\right) \\ &= \left(\frac{M}{\frac{4}{3}R^3}\right)\left(\frac{4}{3}r^3\right) \\ &= M\left(\frac{r^3}{R^3}\right). \end{aligned}$$

Consequently, the gravitational force attracting the particle to the center of the earth is

$$\begin{aligned} F_r &= \frac{-GM_r m}{r^2} \\ &= \frac{-Gm}{r^2}\left[M\left(\frac{r^3}{R^3}\right)\right] \\ &= \frac{-GMm}{R^3} r. \end{aligned}$$

Apply Newton's second law to get the equation of motion for a mass that oscillates through the center of the earth.

$$\sum F = ma$$

The only force acting on the mass is the gravitational force.

$$F_r = ma$$

Substitute the formula for F_r and use the fact that acceleration is the second derivative of position.

$$\frac{-GMm}{R^3}y = m\frac{d^2y}{dt^2}$$

Divide both sides by m.

$$\frac{d^2y}{dt^2} = -\frac{GM}{R^3}y$$

Since the mass is dropped from rest at the earth's surface, the initial conditions associated with this ODE are $y(0) = R$ and $y'(0) = 0$. Set $k^2 = GM/R^3$.

$$y'' = -k^2y$$

This is a linear homogeneous ODE with constant coefficients, so it has solutions of the form $y = e^{pt}$.

$$y = e^{pt} \quad \rightarrow \quad y' = pe^{pt} \quad \rightarrow \quad y'' = p^2e^{pt}$$

Substitute these formulas into the ODE.

$$p^2e^{pt} = -k^2(e^{pt})$$

Divide both sides by e^{pt}.

$$p^2 = -k^2$$

Solve for p.

$$p = \{-ik, ik\}$$

Two solutions to the ODE are $y = e^{-ikt}$ and $y = e^{ikt}$. According to the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} y(t) &= C_1e^{-ikt} + C_2e^{ikt} \\ &= C_1(\cos kt - i\sin kt) + C_2(\cos kt + i\sin kt) \\ &= (C_1 + C_2)\cos kt + (-iC_1 + iC_2)\sin kt \\ &= C_3\cos kt + C_4\sin kt \end{aligned}$$

Differentiate it with respect to t.

$$y'(t) = -C_3k\sin kt + C_4k\cos kt$$

Apply the initial conditions to determine C_3 and C_4.

$$y(0) = C_3 = R$$

$$y'(0) = C_4k = 0$$

Solving this system yields $C_3 = R$ and $C_4 = 0$. Therefore,

$$y(t) = R\cos kt.$$

The period of the particle is

$$T = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\frac{GM}{R^3}}} = 2\pi\sqrt{\frac{R^3}{GM}}.$$

Notice that the particle reaches the center of the earth when $y(t) = 0$, or when $kt = \pi/2$. To find the speed that the particle has as it goes through the center of the earth, evaluate $\left|y'\left(\frac{\pi}{2k}\right)\right|$.

$$y'(t) = -Rk\sin kt \quad \Rightarrow \quad \left|y'\left(\frac{\pi}{2k}\right)\right| = Rk = R\sqrt{\frac{GM}{R^3}} = \sqrt{\frac{GM}{R}}$$

The constants have the following numerical values.

$$M = 5.9736 \times 10^{24} \text{ kg}$$

$$R = 6.378136 \times 10^{6} \text{ m}$$

$$G = 6.67384 \times 10^{-11} \ \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}$$

That means the period and speed are

$$T \approx 5068.91 \not{\text{s}} \times \frac{1 \text{ min}}{60 \not{\text{s}}} \approx 84.5 \text{ minutes}$$

$$\left|y'\left(\frac{\pi}{2k}\right)\right| \approx 7906.04 \ \frac{\not{\text{m}}}{\not{\text{s}}} \times \frac{3.28 \not{\text{ft}}}{1 \not{\text{m}}} \times \frac{1 \text{ mi}}{5280 \not{\text{ft}}} \times \frac{3600 \not{\text{s}}}{1 \text{ h}} \approx 17\,681 \ \frac{\text{mi}}{\text{h}}.$$